

Quantum chaos in many-particle systems

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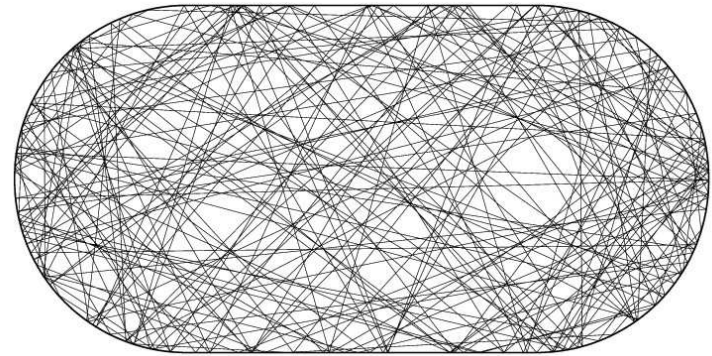
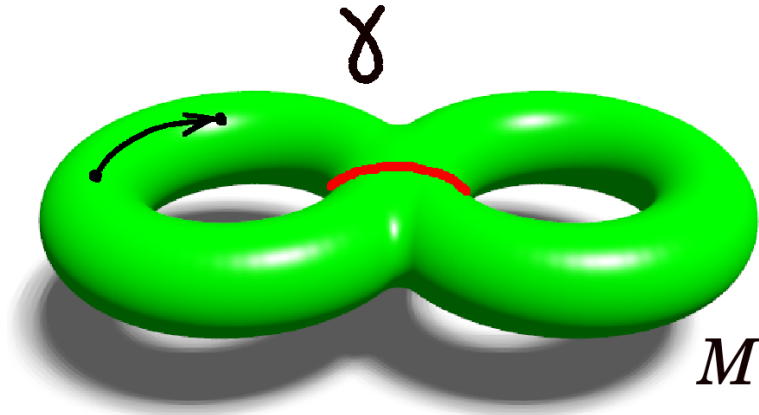
Outline of the talk

- **“Single”-particle quantum chaos.**
Single (semiclassical) limit: $\hbar \rightarrow 0$

- **Many-particle quantum chaos.**
Double limit: $N \rightarrow \infty, \hbar \rightarrow 0$

B.G. & V. AI. Osipov, Nonlinearity 29 (2016)
arXiv:1503.02676

Chaos & Spectral universality



Classical chaos: $\delta(t) \sim \delta(0)e^{\lambda t}$

Motivation

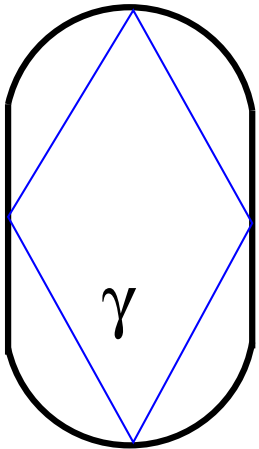
Quantum: $-\Delta\varphi_n = \lambda_n\varphi_n, \quad \varphi_n \in L^2(M)$

BGS conjecture G.Casati, et al. 1980; O. Bohigas, et al. 1984: Correlations of $\{\lambda_n\}_{n=1}^{\infty}$ are universal, described by Random Matrix Ensembles from the same symmetry class

Semiclassical approach

Gutzwiller's trace formula:

$$\rho(E) = \sum_n \delta(E - E_n) \sim \underbrace{\bar{\rho}(E)}_{\text{Smooth}} + \underbrace{\Re \sum_{\gamma \in \text{PO}} \mathcal{A}_\gamma \exp\left(\frac{i}{\hbar} S_\gamma(E)\right)}_{\text{Oscillating}}$$



\mathcal{A}_γ stability factor,
 $S_\gamma(E)$ action of a **periodic orbit** γ

Number of periodic orbits grows exponentially with length

- No prediction on E_n from an individual γ
- **All $\{\gamma\}$ together \iff spectrum**

Two-point correlation function

$$R(\varepsilon) = \frac{1}{\bar{\rho}^2} \langle \rho(E + \varepsilon/\bar{\rho}) \rho(E) \rangle_E - 1$$

$$K(\tau) = \int_{-\infty}^{+\infty} R(\varepsilon) e^{-2\pi i \tau \varepsilon} d\varepsilon \approx \text{(Semiclassically)}$$

$$\approx \frac{1}{T_H^2} \left\langle \sum_{\gamma, \gamma'} \mathcal{A}_\gamma \mathcal{A}_{\gamma'}^* e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta \left(\tau - \frac{(T_\gamma + T_{\gamma'})}{2T_H} \right) \right\rangle_E,$$

$T_\gamma, T_{\gamma'}$ are periods of γ, γ' , $T_H = 2\pi\hbar\bar{\rho}$ (Heisenberg time)

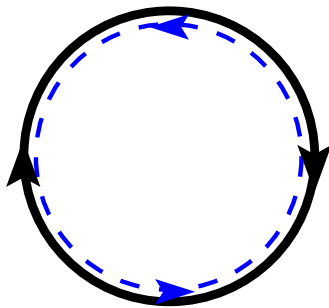
Spectral correlations \iff

Correlations between actions of periodic orbits

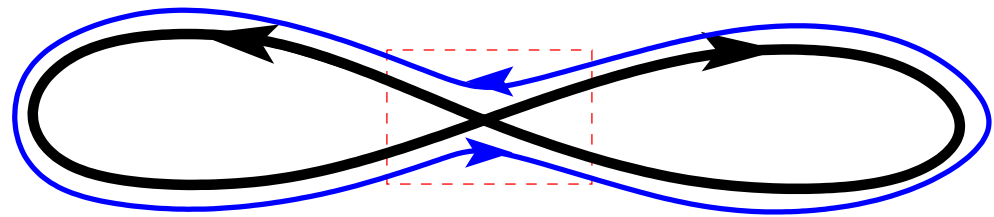
Classical origins of universality

$$K(\tau) = c_1\tau + c_2\tau^2 \dots$$

c_1 – diagonal approximation $\gamma = \gamma'$ M. Berry 1985



Diagonal approximation



Sieber–Richter pairs

c_2 – non-trivial correlations (Sieber-Richter pairs)

M. Sieber K. Richter 2001

$S_\gamma - S_{\gamma'} \sim \hbar \implies$ Duration of encounter $\sim \underbrace{\tau_E = \lambda^{-1} |\log \hbar|}_{\text{Ehrenfest time}}$

All orders in $\tau =$ **RMT result** S. Müller, et. al., 2004

Symbolic Dynamics

Continues flow \implies Map T (Poincare section)

\mathfrak{p}

0	1
\vdots	\vdots
$l-2$	$l-1$

\mathfrak{q}

Phase space partition:

$$V = V_0 \cup V_1 \cup \dots \cup V_{l-1}$$

Point in the phase space:

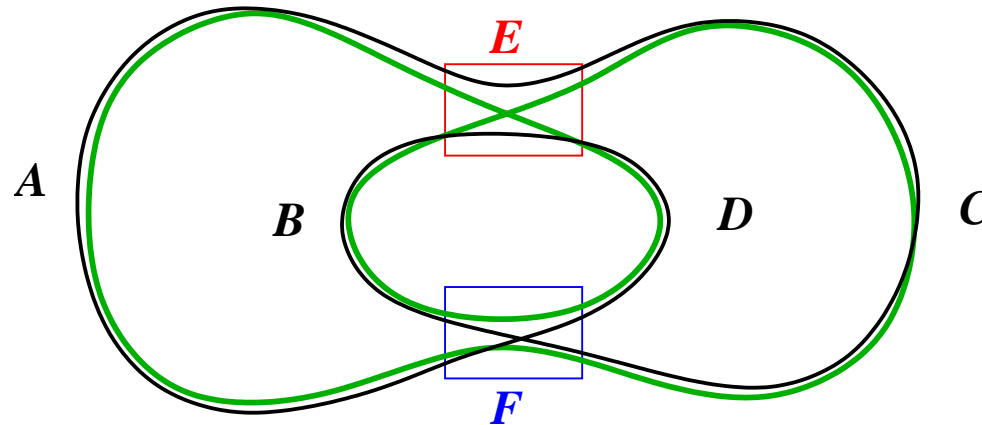
$$x = \underbrace{\dots x_{-1}x_0}_{\text{past}} \cdot \underbrace{x_1x_2\dots}_{\text{future}}; \quad x_i \in \underbrace{\{0, 1, \dots, l-1\}}_{\text{alphabet}}$$

$$Tx = \dots x_{-1}x_0x_1 \cdot x_2x_3 \dots$$

Periodic orbits $\iff [x_1x_2 \dots x_n]$

Partner orbits

B. G, V. Osipov 2013



$$[\gamma_1] = [AECFBEDF], \quad [\gamma_2] = [AEDFBECF]$$

$$E = e_1 e_2 \dots e_p, \quad F = f_1 f_2 \dots f_p$$

Each p-subsequence of symbols from γ_1 appears in γ_2
Locally similar but not identical \implies

Two orbits pass approximately the same points of the phase space:

$$\|\gamma_1 - \gamma_2\| \sim \Lambda^{-p}$$

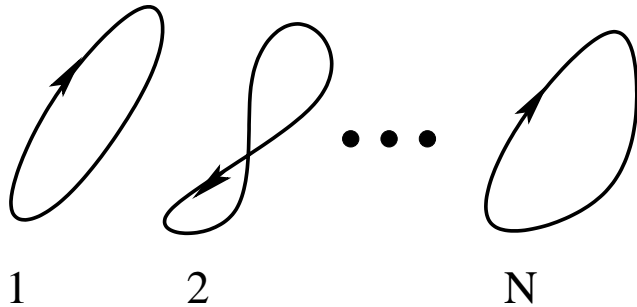
Many-particle systems

$$\mathcal{H} = \sum_{n=1}^N \frac{p_n^2}{2m} + V(x_n) + V_{\text{int}}(x_n - x_{n+1})$$

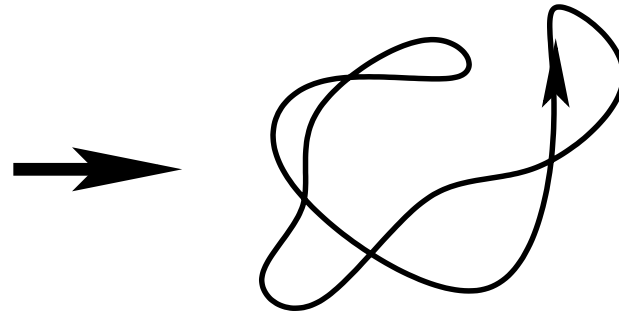
Chaos, Local interactions, Invariance under $n \rightarrow n + 1$

Two views on dynamics:

Many-particle Periodic Orbit
d-dimensions



Single-particle Periodic Orbit
Nd-dimensions

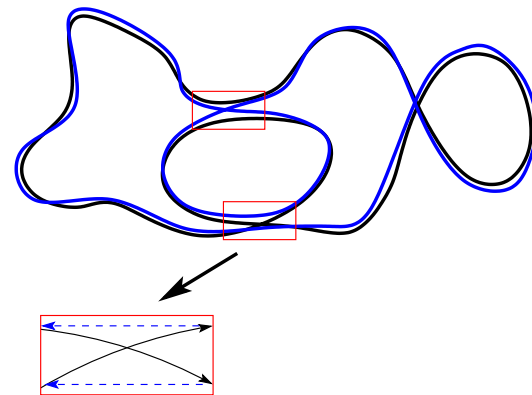
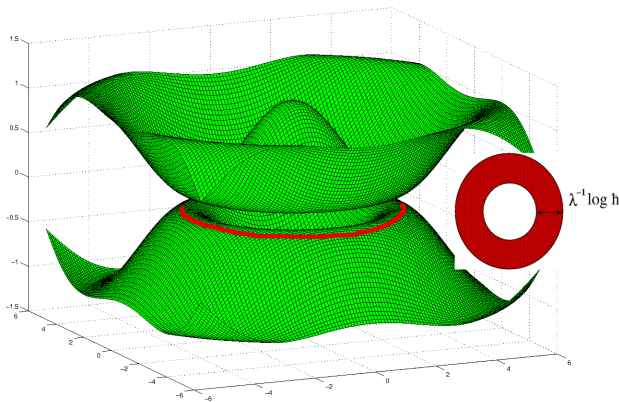


Q: Is the single-particle theory of Quantum Chaos applicable?

Semiclassical “Field Theory”

Continuous limit: $n \rightarrow \eta \in [0, \ell]$, $x_{n,t} \rightarrow \phi(\eta, t)$

$$\mathcal{L} = \sum_{n=1}^N \frac{\dot{q}_{n,t}^2}{2m} + \kappa(x_{n,t} - x_{n+1,t})^2 - V(x_{n,t}) \implies$$
$$\mathcal{L} = \int_0^\ell d\eta (\partial_t \phi(\eta, t))^2 + (\partial_\eta \phi(\eta, t))^2 - V(\phi(\eta, t))$$



- 1) **PO** -are **2D toric surfaces** in d -dim space (Rather than 1D lines in $N \cdot d$ -dim)
- 2) **Encounters** are “**rings**” (Rather than 1D stretches) of “width” $\sim \lambda^{-1} |\log \hbar_{eff}|$

2D Symbolic Dynamics

4
1
3
1
2
1
2
3
4
3
2
4
3
1
2
1
3
1

T

T	2	4	1	2	1	4	2	1	3	4	2	3	2	3	4	3
	2	3	1	2	1	3	4	2	3	1	2	3	1	4	2	3
	3	2	1	4	3	4	1	1	1	3	1	4	1	2	1	1
	2	4	2	1	4	2	4	3	3	1	2	4	4	1	2	3
	4	1	4	3	2	4	2	1	4	4	1	4	3	4	1	2
	3	4	3	1	3	2	4	2	3	1	2	1	4	2	4	4
	1	1	2	4	3	1	3	2	3	1	4	3	3	4	2	4
	1	4	2	3	1	4	2	4	4	4	3	4	2	2	4	3
	3	2	3	1	4	1	4	1	2	1	2	4	3	1	3	1
	1	4	2	2	3	2	1	3	4	4	2	3	1	4	2	3
	3	1	1	4	1	4	3	2	1	3	2	2	3	3	1	4

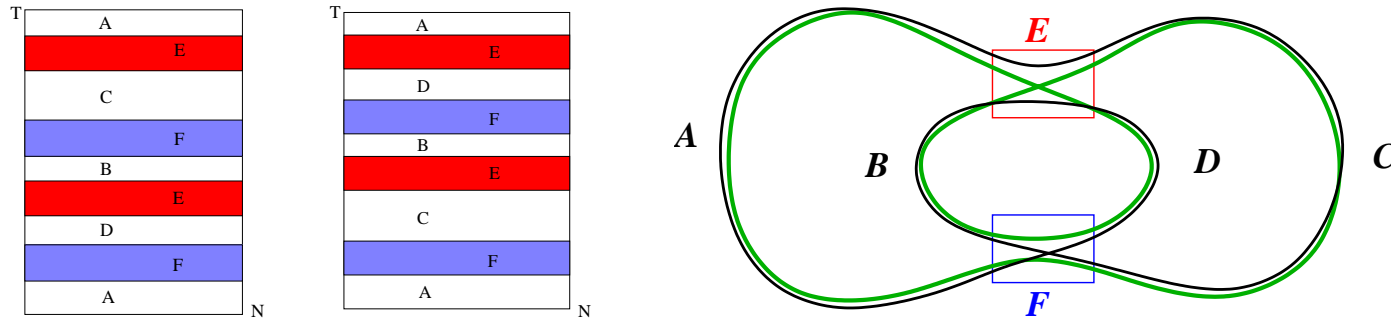
N

- 1) **Small alphabet** (does not grow with N)
- 2) **Uniqueness:** Each PO Γ is uniquely encoded by \mathbb{M}_Γ
- 3) **Locality:** $r \times r$ square of symbols around (n, t) defines position of the n 'th particle at the time t up to error $\sim \Lambda^{-r}$

Encounter - repeating region of symbols

Different types of Partner Orbits

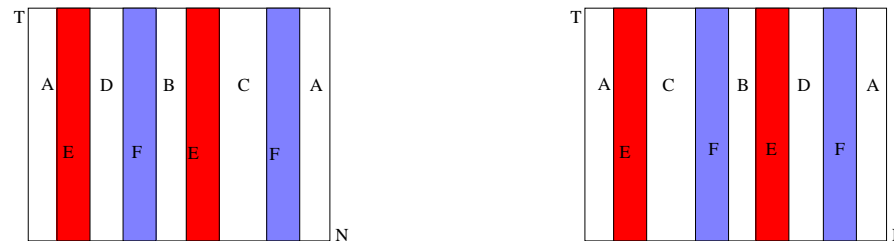
A. Single particle partners:



Dominant iff $T \gtrsim W_{\hbar} \gtrsim N$ - **Single particle theory**

$$W_{\hbar} \sim \Lambda^{-1} |\log \hbar_{eff}| \approx \text{Width of encounter}$$

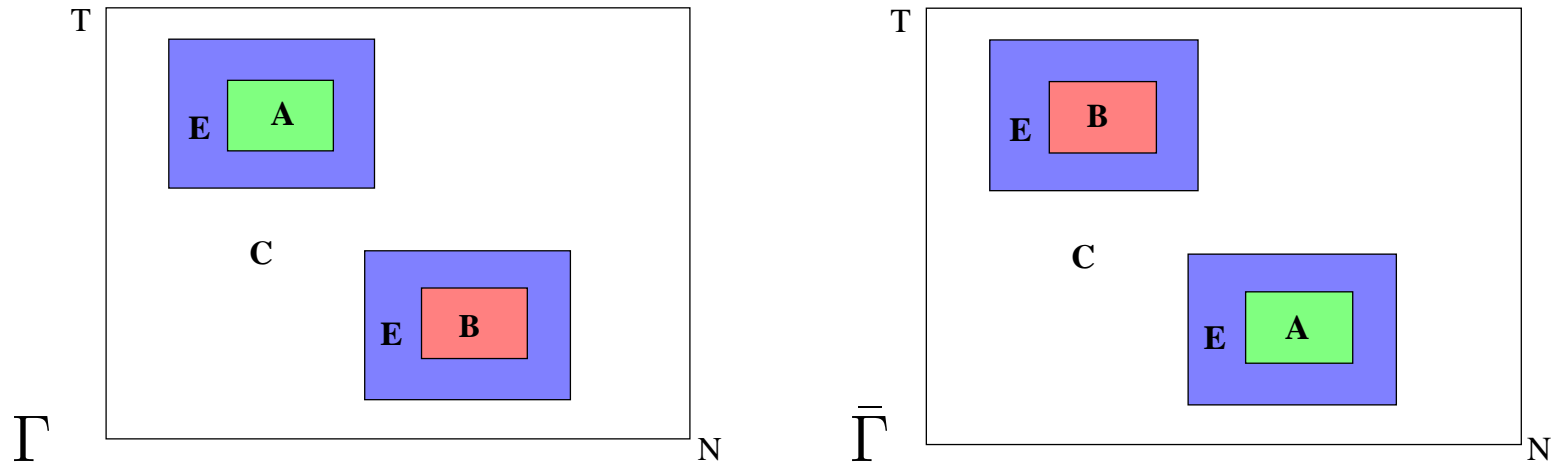
B. Dual partners:



Dominant iff $T \lesssim W_{\hbar} \lesssim N$ - **Thermodynamic, short time regime**

Different types of Partner Orbits

C. If $T \gtrsim W_{\hbar}$, $N \gtrsim W_{\hbar}$ i.e. T and N are larger then “Ehrenfest scale”:

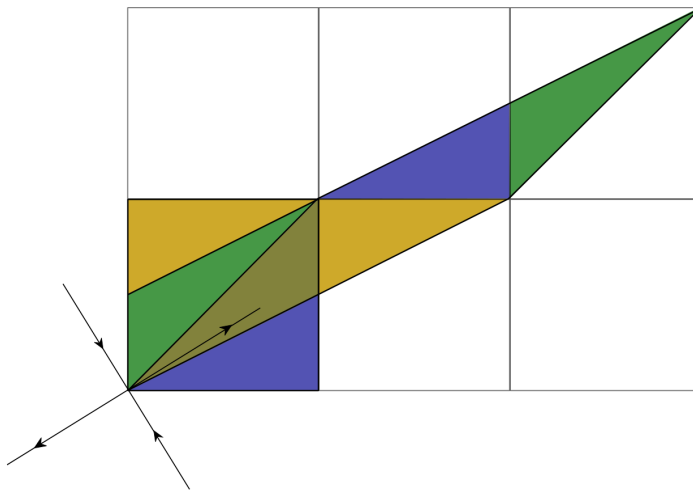


Note: One encounter is enough, even if time reversal symmetry is broken

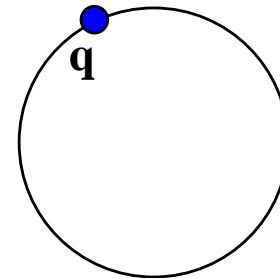
B, C - Genuine many-particle Quantum Chaos!

A Lone Cat Map: $\mathbb{T}^2 \rightarrow \mathbb{T}^2$

Phase space: $q_t, p_t \in [0, 1)$, **windings** $m_t = (m_t^q, m_t^p) \in \mathbb{Z}$



Configuration Space

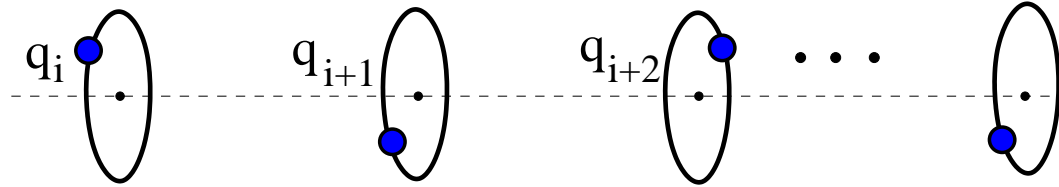


$$\begin{pmatrix} q_{t+1} \\ p_{t+1} \end{pmatrix} = \begin{pmatrix} a & 1 \\ ab - 1 & b \end{pmatrix} \begin{pmatrix} q_t \\ p_t \end{pmatrix} - \begin{pmatrix} m_t^q \\ m_t^p \end{pmatrix},$$

$a, b \in \mathbb{Z}$. Chaos if $|a + b| > 2$

Newton form: $\Delta q_t \equiv q_{t+1} - 2q_t + q_{t-1} = (a + b - 2)q_t - m_t$

Coupled-Cat Maps: $T^{2N} \rightarrow T^{2N}$



$$S(\mathbf{q}_t, \mathbf{q}_{t+1}) = S_0(\mathbf{q}_t, \mathbf{q}_{t+1}) + S_{\text{int}}(\mathbf{q}_t), \quad \mathbf{q}_t = (q_{1,t}, q_{2,t} \cdots q_{N,t})$$

N Interacting cat maps, $q_{n,t}, p_{n,t} \in [0, 1)$:

$$S_0 = \sum_{n=1}^N S_{\text{cat}}(q_{n,t}, q_{n,t+1}) + V(q_{n,t}); \quad S_{\text{int}} = \underbrace{- \sum_{n=1}^N q_{n,t} q_{1+n,t}}_{\text{interactions}}$$

Equations of motion:

$$p_{n,t} = - \frac{\partial S}{\partial q_{n,t}} \quad p_{n,t+1} = \frac{\partial S}{\partial q_{n,t+1}}$$

Classical Particle-time Duality

Newtonian form:

$$\Delta q_{n,t} = (a + b - 4)q_{n,t} + V'(q_{n,t}) - m_{n,t}$$

Discrete Laplacian:

$$\Delta f_{n,t} \equiv f_{n+1,t} + f_{n-1,t} + f_{n,t+1} + f_{n,t-1} - 4f_{n,t}$$

Particle-time symmetry: $t \longleftrightarrow n \implies$

**N -particle POs $\{\Gamma\}$ of period $T \iff T$ -particle POs $\{\Gamma'\}$
of period N**

$$S(\Gamma) = S(\Gamma'), \quad A_{\Gamma} = A_{\Gamma'}$$

$\{m_{n,t}\}$ - provide symbolic encoding of POs

2D Symbolic Dynamics

T	2	4	1	2	1	4	2	1	3	4	2	3	2	3	4	3
	2	3	1	2	1	3	4	2	3	1	2	3	1	4	2	3
	3	2	1	4	3	4	1	1	1	3	1	4	1	2	1	1
	2	4	2	1	4	2	4	3	3	1	2	4	4	1	2	3
	4	1	4	3	2	4	2	1	4	4	1	4	3	4	1	2
	3	4	3	1	3	2	4	2	3	1	2	1	4	2	4	4
	1	1	2	4	3	1	3	2	3	1	4	3	3	4	2	4
	1	4	2	3	1	4	2	4	4	4	3	4	2	2	4	3
	3	2	3	1	4	1	4	1	2	1	2	4	3	1	3	1
	1	4	2	2	3	2	1	3	4	4	2	3	1	4	2	3
	3	1	1	4	1	4	3	2	1	3	2	2	3	3	1	4
																N

$$\mathbb{M}_\Gamma = \begin{pmatrix} m_{1,1} & m_{2,1} & \dots & m_{N,1} \\ m_{1,2} & m_{2,2} & \dots & m_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1,T} & m_{2,T} & \dots & m_{N,T} \end{pmatrix}$$

- ✓ **Small alphabet** (does not grow with N)
 - ✓ **Uniqueness** + Γ can be easily restored from \mathbb{M}_Γ
 - ✓ **Locality** ($r \times r$ square of symbols around (n, t) defines approx. position of the n 'th particle at the time t)
- B.G. V. Osipov (2015),
 B.G., L Han, R. Jafari, A. K. Saremi, P Cvitanović (2016)

Example of Partner Orbits

$$T = 50, N = 70, a = 3, b = 2$$

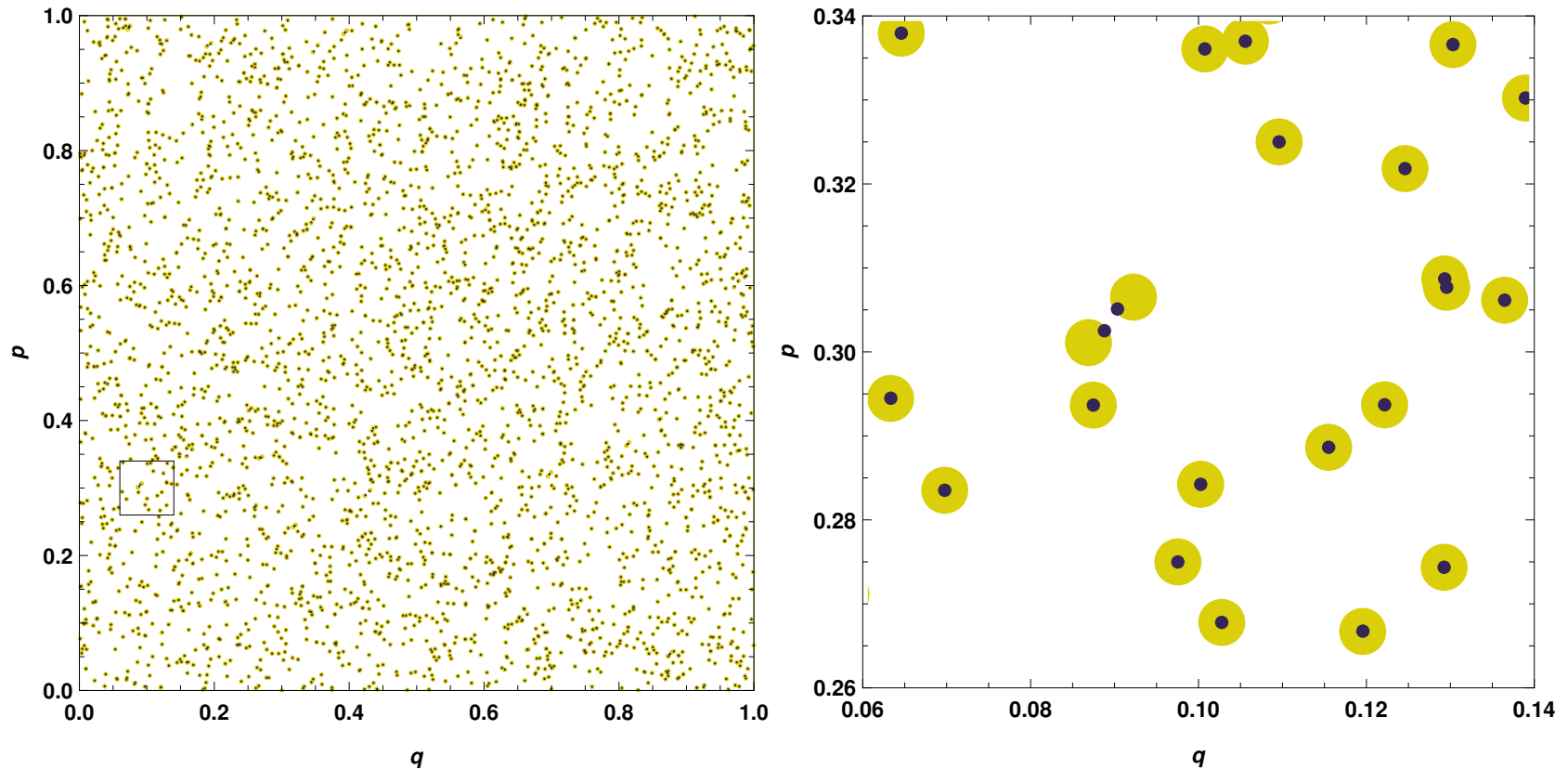
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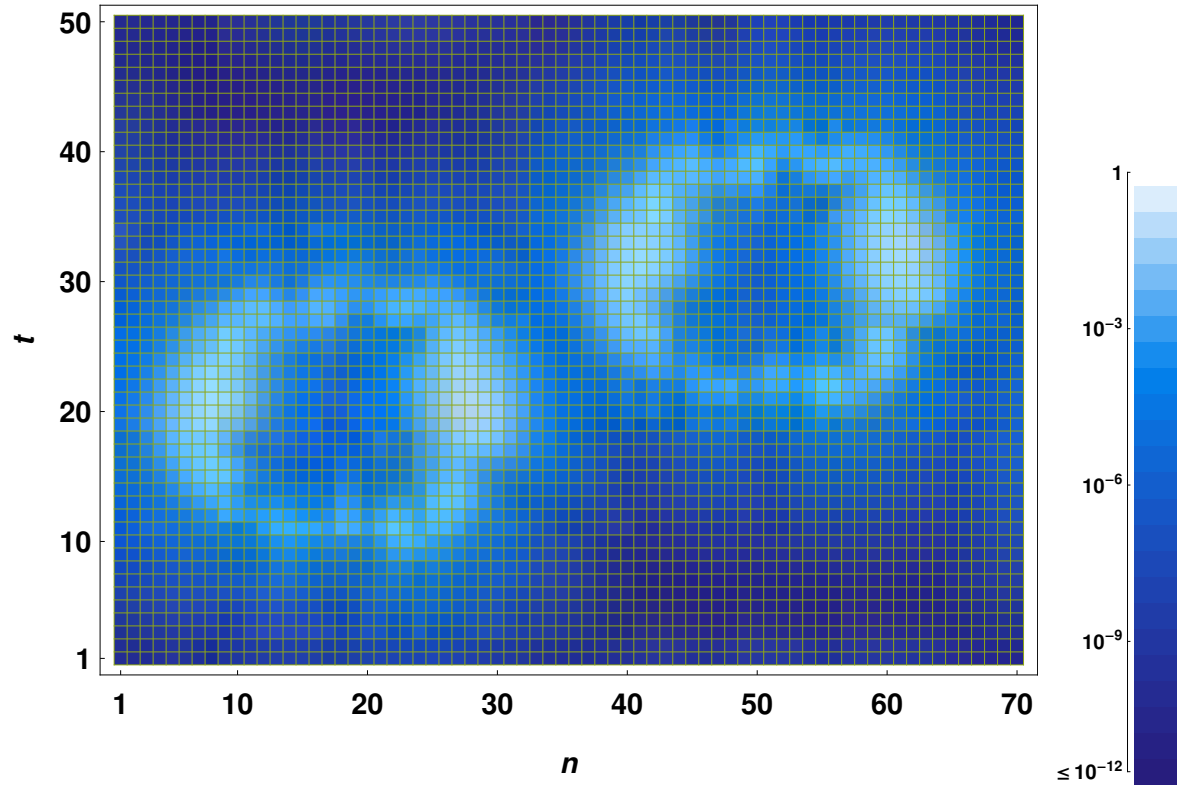
m	(0)	(0)	(1)	(1)	(1)	(2)	(2)	(2)	(3)	(3)	(0)	(-1)	(-1)	(-1)	(-2)	(-2)
a	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example of Partner Orbits



All the points of $\Gamma = \{(q_{n,t}, p_{n,t})\}$ and $\bar{\Gamma} = \{(\bar{q}_{n,t}, \bar{p}_{n,t})\}$ are paired

Distances between paired points



$$d_{n,t} = \sqrt{(q_{n,t} - \bar{q}_{n',t'})^2 + (p_{n,t} - \bar{p}_{n',t'})^2},$$

Largest distances $\sim 2 \cdot 10^{-3}$ are between points in encounters

Quantisation

Hannay, Berry (1980); Keating (1991)

U_N is $L^N \times L^N$ unitary matrix, $L = \hbar_{eff}^{-1}$

Translational symmetries: $\implies N$ subspectra
approximately of the same size $= L^N / N$

Gutzwiller trace formula

Rivas, Saraceno, A. de Almeida (2000)

$$\text{Tr} (U_N)^T = |\det(\mathcal{B}_N^T - 1)|^{-\frac{1}{2}} \sum_{\Gamma \in \text{PO}} \exp(-i2\pi L S_\Gamma).$$

All entries are symmetric under exchange $N \leftrightarrow T$

Quantum Duality

$$\text{Tr} (U_N)^T = \text{Tr} (U_T)^N$$

$$\text{Form Factor: } K_N(T) = \frac{1}{2L^N} \left\langle \left| \text{Tr} (U_N)^T \right|^2 \right\rangle$$

For short times $T < n_E = \lambda^{-1} \log L$, $N \sim L^T$

Regime dual to universal:

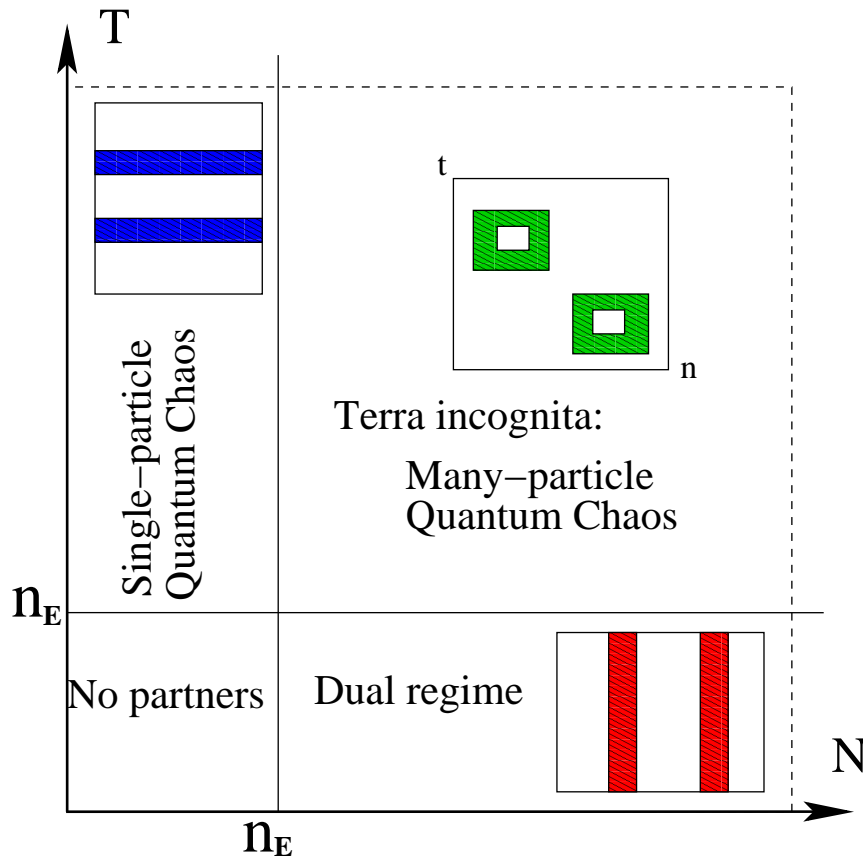
$$K_N(T) = L^{T-N} K_\beta(TN/L^T)$$

In particular for very short times $L^T/T < N$, $K_\beta \approx 1$

$$K_N(T) \approx L^T/L^N$$

Short time exponential growth instead of linear TN/L^N

Summary



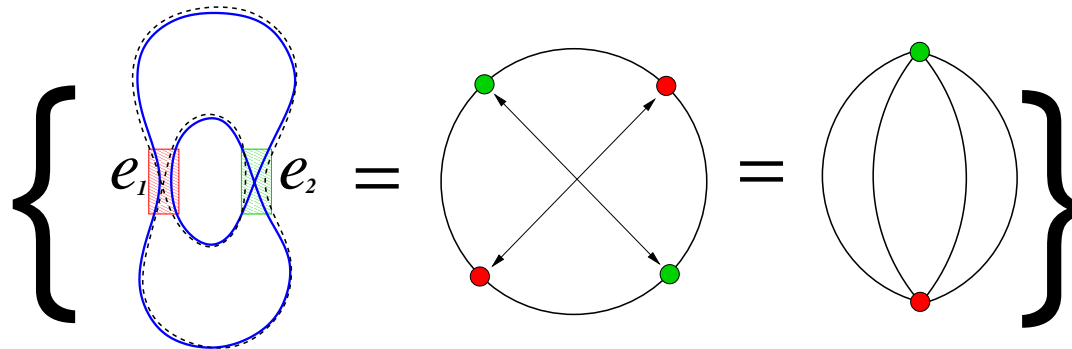
$$\mathcal{K} = \frac{1}{TN} \left\langle \left| \text{Tr} (U_N)^T \right|^2 \right\rangle$$

Duality:

$$\mathcal{K}(N, T) = \mathcal{K}(T, N)$$

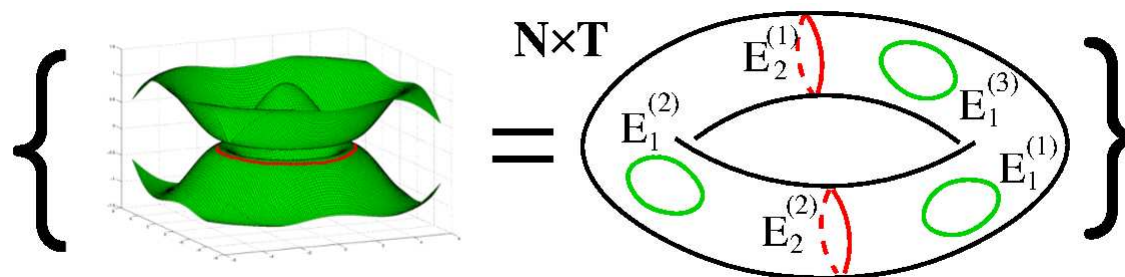
Many-particle Semiclassical Programm

Single-particle structure diagrams:



Distinguished by order of encounters

Many-particle structure diagrams:



Distinguished by order and winding numbers ω of encounters!